Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Approximation Theory 127 (2004) 155–177
Journal of Approximation Theory
http://www.elsevier.com/locate/jat

Estimates for the first and second derivatives of the Stieltjes polynomials

H.S. Jung

Division of Applied Mathematics, KAIST, 373-1 Kusongdong, Yusongku, Taejon 305-701, South Korea

Received 12 March 2003; accepted in revised form 28 February 2004

Communicated by Mourad Ismail

Abstract

Let $w_\lambda(x) := (1-x^2)^{\lambda-1/2}$ and $P_n^{(\lambda)}$ be the ultraspherical polynomials with respect to $w_\lambda(x)$. Then we denote $E_{n+1}^{(\lambda)}$ the Stieltjes polynomials with respect to $w_\lambda(x)$ satisfying

$$\int_{-1}^{1} w_\lambda(x) P_n^{(\lambda)}(x) E_{n+1}^{(\lambda)}(x) x^m \, dx \begin{cases} =0, & 0 \leqslant m < n+1, \\ \neq 0, & m = n+1. \end{cases}$$

In this paper, we give estimates for the first and second derivatives of the Stieltjes polynomials $E_{n+1}^{(\lambda)}$ and the product $E_{n+1}^{(\lambda)} P_n^{(\lambda)}$ by obtaining the asymptotic differential relations. Moreover, using these differential relations we estimate the second derivatives of $E_{n+1}^{(\lambda)}(x)$ and $E_{n+1}^{(\lambda)}(x) P_n^{(\lambda)}(x)$ at the zeros of $E_{n+1}^{(\lambda)}(x)$ and the product $E_{n+1}^{(\lambda)}(x) P_n^{(\lambda)}(x)$, respectively.

MSC: 42C05; 65D05

Keywords: Stieltjes polynomials; Ultraspherical polynomials; Asymptotic differential relations

1. Introduction

In 1894, Stieltjes introduced a remarkable polynomial sequence $\{E_{n+1}(x)\}$ which satisfies the orthogonality relation

$$\int_{-1}^{1} P_n(x) E_{n+1}(x) x^k \, dx = 0, \quad k = 0, 1, 2, \ldots, n,$$

E-mail address: hsjung@jacobi.kaist.ac.kr.

0021-9045/$ - see front matter
doi:10.1016/j.jat.2004.02.004

where $P_n(x)$ is the nth degree Legendre polynomial. Stieltjes made the conjecture that the zeros of $E_{n+1}(x)$ should all be in $(-1,1)$ and should alternate with those of $P_n(x)$. In 1934, Szegő generalized the problem to integrals with a weight function of the form $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$, $\lambda > -1/2$ and showed that the zeros of the generalized Stieltjes polynomials $E_{n+1}^{(\lambda)}(x)$, satisfying the orthogonality

$$\int_{-1}^{1} w_\lambda(x) P_n^{(\lambda)}(x) E_{n+1}^{(\lambda)}(x) x^k \, dx = 0, \quad k = 0, 1, 2, \ldots, n,$$

where $P_n^{(\lambda)}(x)$ is the nth degree polynomial orthogonal in $(-1,1)$ with respect to $w_\lambda(x)$, are also in $(-1,1)$ and interlace with those of $P_n^{(\lambda)}(x)$ whenever $0 < \lambda \leqslant 2$.

After Szegő's works, in 1964 Kronrod introduced the so called Gauss–Kronrod quadrature formulas,

$$\int_{-1}^{1} w_\lambda(x) f(x) \, dx = \sum_{v=1}^{n} A_{v,n}^{\mathrm{GK}} f(x_{v,n}^{(\lambda)}) + \sum_{\mu=1}^{n+1} B_{\mu,n+1}^{\mathrm{GK}} f(\xi_{\mu,n+1}^{(\lambda)}) + R_{2n+1}^{\mathrm{GK}}(f)$$

for estimating the error of the Gaussian quadrature formula and in 1970 Barrucand [1] observed that the zeros of Stieltjes polynomials are precisely the additional nodes $\xi_{1,n+1}^{(\lambda)}, \ldots, \xi_{n+1,n+1}^{(\lambda)}$ of Gauss–Kronrod quadrature formulas. This induced a lot further study of Stieltjes polynomials and Gauss–Kronrod quadrature formulas. These days, Gauss–Kronrod formulas are used extensively in the numerical integration software(cf. e.g. [16,21]). There are the exhaustive surveys of Gautschi [8] and Monegato [13,15], for the precise definition of Gauss–Kronrod quadrature formulas and an overview about their connection to Stieltjes polynomials.

Recently, several authors [3,5,6,9,18] studied further interesting properties for these Stieltjes polynomials. Ehrich [3] investigated asymptotic representations for $E_{n+1}^{(\lambda)}$ and $E_{n+1}^{(\lambda)\prime}(0 \leqslant \lambda \leqslant 1)$ on closed subsets of $(-1,1)$ and Ehrich and Mastroianni [5,6] gave accurate pointwise bounds of the Stieltjes polynomials $E_{n+1}^{(\lambda)}(0 \leqslant \lambda \leqslant 1)$ on $[-1,1]$ so that they proved that the Lagrange interpolatory process is optimal in the sense that nth Lebesgue constant $\sim \log n$. Several of the problems stated in the survey papers [8,13,15] has been solved in the meantime as is recorded in [4,17,19] not yet covering the recent surprising result [20]. Nevertheless, many questions still remain to be answered. In this paper, we find the asymptotic differential relations of the first and the second order for the Stieltjes polynomials $E_{n+1}^{(\lambda)}(x)(0 \leqslant \lambda \leqslant 1)$ and the product $F_{2n+1}^{(\lambda)} := E_{n+1}^{(\lambda)}(x) P_n^{(\lambda)}(x)(0 \leqslant \lambda \leqslant 1)$. Using these relations, we obtain pointwise upper bounds of $E_{n+1}^{(\lambda)\prime}(x)$, $E_{n+1}^{(\lambda)\prime\prime}(x)$, $F_{2n+1}^{(\lambda)\prime}(x)$, and $F_{2n+1}^{(\lambda)\prime\prime}(x)$. We also estimate the value of $E_{n+1}^{(\lambda)\prime\prime}(x)$ and $F_{2n+1}^{(\lambda)\prime\prime}(x)$ at the zeros of $E_{n+1}^{(\lambda)}(x)$ and $F_{n+1}^{(\lambda)}(x)$, respectively. Moreover, these estimates play important roles in proving that the Lebesgue constants of Hermite–Fejér interpolatory process are optimal $(\sim O(1))$(see [12]).

2. Main results

We first introduce some notations, which we use in the following.

For the ultraspherical polynomials $P_n^{(\lambda)}$, $\lambda \neq 0$, we use the normalization $P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n} = O(n^{2\lambda-1})$. A representation of Stieltjes polynomials $E_{n+1}^{(\lambda)}(x)$ is (cf. [15,22])

$$\frac{\gamma_n^{(\lambda)}}{2} E_{n+1}^{(\lambda)}(\cos\theta) = \alpha_{0,n}^{(\lambda)}\cos(n+1)\theta + \alpha_{1,n}^{(\lambda)}\cos(n-1)\theta + \cdots + \begin{cases} \alpha_{\frac{n}{2},n}^{(\lambda)}\cos\theta, & n \text{ even}, \\ \frac{1}{2}\alpha_{\frac{n+1}{2},n}^{(\lambda)}, & n \text{ odd}, \end{cases} \tag{2.1}$$

where

$$\alpha_{0,n}^{(\lambda)} = f_{0,n}^{(\lambda)} = 1, \quad \sum_{\mu=0}^{v} \alpha_{\mu,n}^{(\lambda)} f_{v-\mu,n}^{(\lambda)} = 0, \quad v = 1, 2, \ldots, \tag{2.2}$$

$$f_{v,n}^{(\lambda)} := \left(1 - \frac{\lambda}{v}\right)\left(1 - \frac{\lambda}{n+v+\lambda}\right) f_{v-1,n}^{(\lambda)}, \quad v = 1, 2, \ldots \tag{2.3}$$

and

$$\gamma_n^{(\lambda)} = \sqrt{\pi}\frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1)} \sim \sqrt{\pi} n^{\lambda-1}. \tag{2.4}$$

We denote the zeros of $P_n^{(\lambda)}$ by $x_{v,n}^{(\lambda)} = \cos\phi_{v,n}^{(\lambda)}$, $v = 1, \ldots, n$, and the zeros of Stieltjes polynomials $E_{n+1}^{(\lambda)}$ by $\xi_{\mu,n+1}^{(\lambda)} = \cos\theta_{\mu,n+1}^{(\lambda)}$, $\mu = 1, \ldots, n+1$. We denote the zeros of $F_{2n+1}^{(\lambda)} := P_n^{(\lambda)} E_{n+1}^{(\lambda)}$ by $y_{v,2n+1}^{(\lambda)} = \cos\psi_{v,2n+1}^{(\lambda)}$, $v = 1, \ldots, 2n+1$. All nodes are ordered by increasing magnitude. We set $\varphi(x) := \sqrt{1-x^2}$ and for any two sequences $\{b_n\}_n$ and $\{c_n\}_n$ of nonzero real numbers(or functions), we write $b_n \lesssim c_n$, if there exists a constant $C > 0$, independent of n (and x) such that $b_n \leqslant C c_n$ for n large enough and write $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. We denote by $\mathscr{P}_n$ the space of polynomials of degree at most n.

Ehrich and Mastroianni [5] proved that for $\mu = 0, 1, \ldots, n+2$ and $v = 0, 1, \ldots, 2n+2$

$$|\theta_{\mu,n+1}^{(\lambda)} - \theta_{\mu+1,n+1}^{(\lambda)}| \sim |\psi_{v,2n+1}^{(\lambda)} - \psi_{v+1,2n+1}^{(\lambda)}| \sim n^{-1}, \tag{2.5}$$

where $\psi_{0,2n+1}^{(\lambda)} := \theta_{0,n+1}^{(\lambda)} := \pi$ and $\psi_{2n+2,2n+1}^{(\lambda)} := \theta_{n+2,n+1}^{(\lambda)} := 0$. Also, they obtained precise upper bounds of the Stieltjes polynomials $E_{n+1}^{(\lambda)}(x)$ such as:

Proposition 2.1 (Ehrich and Mastroianni [5, Theorem 2.1]). *Let* $0<\lambda<1$. *Then for* $n\geqslant 0$,

$$|E_{n+1}^{(\lambda)}(x)| \lesssim n^{1-\lambda}\varphi^{1-\lambda}(x)+1 \quad -1\leqslant x\leqslant 1. \tag{2.6}$$

Furthermore, $E_{n+1}^{(\lambda)}(1)\gtrsim 1$.

The associated sin-polynomial is defined by

$$\begin{aligned}\frac{\gamma_n^{(\lambda)}}{2}e_n^{(\lambda)}(\theta) &= \alpha_{0,n}^{(\lambda)}\sin(n+1)\theta+\alpha_{1,n}^{(\lambda)}\sin(n-1)\theta\\ &\quad +\cdots+\begin{cases}\alpha_{\frac{n}{2}-1,n}^{(\lambda)}\sin 3\theta+\alpha_{\frac{n}{2},n}^{(\lambda)}\sin\theta, & n \text{ even},\\ \alpha_{\frac{n-1}{2},n}^{(\lambda)}\sin 2\theta, & n \text{ odd}\end{cases}\end{aligned} \tag{2.7}$$

and is important in connection with polynomials $G_n^{(\lambda)}$ considered by Geronimus (cf. [10,15,18,22]). The connection is (cf. [18,22])

$$\sin\theta G_n^{(\lambda)}(\cos\theta) = e_n^{(\lambda)}(\theta).$$

First, we state the asymptotic differential relations of the first order for $E_{n+1}^{(\lambda)}(\cos\theta)$ and $e_n^{(\lambda)}(\theta)$.

Proposition 2.2. *Let* $0<\lambda<1$. *Then for* $0\leqslant\theta\leqslant\pi$,

$$\frac{d}{d\theta}E_{n+1}^{(\lambda)}(\cos\theta) = -(n+1)e_n^{(\lambda)}(\theta)+p_1(\theta) \tag{2.8}$$

and

$$\frac{d}{d\theta}e_n^{(\lambda)}(\theta) = (n+1)E_{n+1}^{(\lambda)}(\cos\theta)+p_2(\theta), \tag{2.9}$$

where $\tilde{p}_1(x) := p_1(\theta)/\sin\theta\in\mathscr{P}_{n-2}$, $\tilde{p}_2(x) := p_2(\theta)\in\mathscr{P}_{n-1}$ $(x=\cos\theta)$, *and satisfy that*

$$\max_{0\leqslant\theta\leqslant\pi}\{|p_1(\theta)|,|p_2(\theta)|\}\lesssim n. \tag{2.10}$$

If we restate (2.8) by $G_n^{(\lambda)}(x)$, then

$$\frac{d}{dx}E_{n+1}^{(\lambda)}(x) = (n+1)G_n^{(\lambda)}(x)-\tilde{p}_1(x),$$

where $\tilde{p}_1(x)\in\mathscr{P}_{n-2}$ and satisfies $|\tilde{p}_1(x)|\lesssim n/\sqrt{1-x^2}$ on $(-1+1/n,1-1/n)$ and $|\tilde{p}_1(x)|\lesssim n^2$ on $(-1,-1+1/n]\cup[1-1/n,1)$.

In the following, we state the asymptotic differential relation of the second order of $E_{n+1}^{(\lambda)}$.

Proposition 2.3. *Let* $0<\lambda<1$. *Then for all* $x\in[-1,1]$,

$$(1-x^2){E_{n+1}^{(\lambda)}}''(x)-x{E_{n+1}^{(\lambda)}}'(x)+(n+1)^2E_{n+1}^{(\lambda)}(x) = I_n(x), \tag{2.11}$$

where $I_n \in \mathscr{P}_{n-1}$ *and satisfies that*

$$\max_{x\in[-1,1]} |I_n(x)| \lesssim n^2. \tag{2.12}$$

The form of the left-hand side in (2.11) is the second-order differential form for Chebyshev polynomials so that the Stieltjes polynomials $E_{n+1}^{(\lambda)}$ can be treated similarly to Chebyshev polynomials with the error terms $I_n(x)$.

Next, using Propositions 2.2 and 2.3, we obtain bounds of the associated sin-polynomial $e_n^{(\lambda)}(\theta)$.

Theorem 2.4. *Let* $0<\lambda<1$. *Then, for all* $\theta\in[0,\pi]$,

$$|e_n^{(\lambda)}(\theta)| \lesssim n^{1-\lambda}\varphi^{1-\lambda}(\cos\theta)+1. \tag{2.13}$$

From Propositions 2.2, 2.3, and Theorem 2.4, we can obtain bounds of the first and second derivatives of $E_{n+1}^{(\lambda)}$.

Theorem 2.5. *Let* $0<\lambda<1$. (a) *For all* $x\in[\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,

$$|E_{n+1}^{(\lambda)\prime}(x)| \lesssim n^{2-\lambda}\varphi^{-\lambda}(x). \tag{2.14}$$

Moreover, we have for $x\in[-1,\xi_{1,n+1}^{(\lambda)}]\cup[\xi_{n+1,n+1}^{(\lambda)},1]$,

$$|E_{n+1}^{(\lambda)\prime}(x)| \sim n^2. \tag{2.15}$$

(b) *For all* $x\in[\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,

$$|E_{n+1}^{(\lambda)\prime\prime}(x)| \lesssim n^{3-\lambda}\varphi^{-1-\lambda}(x). \tag{2.16}$$

Moreover, we have for $x\in[-1,\xi_{1,n+1}^{(\lambda)}]\cup[\xi_{n+1,n+1}^{(\lambda)},1]$,

$$|E_{n+1}^{(\lambda)\prime\prime}(x)| \sim n^4. \tag{2.17}$$

Inequalities (2.14) for $E_{n+1}^{(\lambda)\prime}(x)$ are given for the first time in the paper of Monegato [14] and can be obtained for interval of the form $[-1+\varepsilon, 1-\varepsilon]$, $\varepsilon>0$ by the asymptotic relation for the first derivative of $E_{n+1}^{(\lambda)}(x)$ given in [3, Theorem].

We also obtain the asymptotic differential relations of the second order for $F_{2n+1}^{(\lambda)}(x)$.

Corollary 2.6. *Let* $0<\lambda<1$. *Then for all* $x\in[-1,1]$,

$$(1-x^2)F_{2n+1}^{(\lambda)\prime\prime}(x) - xF_{2n+1}^{(\lambda)\prime}(x) + (2n^2+2(1+\lambda)n+1)F_{2n+1}^{(\lambda)}(x) = J_n(x), \tag{2.18}$$

where $J_n(x) \in \mathscr{P}_{2n+1}$ *and satisfies that for* $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$

$$|J_n(x)| \lesssim n^2 \varphi^{1-2\lambda}(x) \tag{2.19}$$

and for $x \in [-1, \xi_{1,n+1}^{(\lambda)}] \cup [\xi_{n+1,n+1}^{(\lambda)}, 1]$

$$|J_n(x)| \lesssim n^{1+2\lambda}.$$

Corollary 2.7. *Let* $0 < \lambda < 1$. (a) *For all* $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,

$$|F_{2n+1}^{(\lambda)}{}'(x)| \lesssim n\varphi^{-2\lambda}(x). \tag{2.20}$$

Moreover, we have for $x \in [-1, \xi_{1,n+1}^{(\lambda)}] \cup [\xi_{n+1,n+1}^{(\lambda)}, 1]$,

$$|F_{2n+1}^{(\lambda)}{}'(x)| \sim n^{1+2\lambda}. \tag{2.21}$$

(b) *For all* $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,

$$|F_{2n+1}^{(\lambda)}{}''(x)| \lesssim n^2 \varphi^{-1-2\lambda}(x). \tag{2.22}$$

Moreover, we have for $x \in [-1, \xi_{1,n+1}^{(\lambda)}] \cup [\xi_{n+1,n+1}^{(\lambda)}, 1]$,

$$|F_{2n+1}^{(\lambda)}{}''(x)| \sim n^{3+2\lambda}. \tag{2.23}$$

For $[-1+\varepsilon, 1-\varepsilon]$, $\varepsilon > 0$ inequalities (2.20) can be also obtained by the asymptotic representations in [3, Theorem].

For the first derivatives, the following results are obtained in [5, Lemma 5.5].

Proposition 2.8 (Ehrich and Mastroianni [5, Lemma 5.5]). *Let* $0 < \lambda < 1$. *Then for* $\mu = 1, 2, \ldots, n+1$,

$$|E_{n+1}^{(\lambda)}{}'(\xi_{\mu,n+1}^{(\lambda)})| \sim n^{2-\lambda} \varphi^{-\lambda}(\xi_{\mu,n+1}^{(\lambda)}) \tag{2.24}$$

and for $v = 1, 2, \ldots, 2n+1$,

$$|F_{2n+1}^{(\lambda)}{}'(y_{v,2n+1}^{(\lambda)})| \sim n\varphi^{-2\lambda}(y_{v,2n+1}^{(\lambda)}). \tag{2.25}$$

We now estimate the second derivatives at the zeros of $E_{n+1}^{(\lambda)}$ and $F_{2n+1}^{(\lambda)}$.

Theorem 2.9. *Let* $0 < \lambda < 1$. *Then for* $\mu = 1, 2, \ldots, n+1$,

$$|E_{n+1}^{(\lambda)}{}''(\xi_{\mu,n+1}^{(\lambda)})| \lesssim n^2 \varphi^{-2}(\xi_{\mu,n+1}^{(\lambda)}) \tag{2.26}$$

and for $v = 1, 2, \ldots, 2n+1$,

$$|F_{2n+1}^{(\lambda)}{}''(y_{v,2n+1}^{(\lambda)})| \lesssim n^{1+\lambda} \varphi^{-2-\lambda}(y_{v,2n+1}^{(\lambda)}). \tag{2.27}$$

Considering (2.16) and (2.22), estimates (2.26) and (2.27) are remarkable. In fact, (2.26) and (2.27) play key roles in proving that the Lebesgue constant of Hermite–Fejér interpolation operators are optimal(see [12]), i.e. for $0<\lambda<1$

$$||H_{n+1}|| := \sup_{||f||_\infty=1} ||H_{n+1}[f]||_\infty = O(1),$$

and for $0<\lambda\leqslant 1/2$

$$||\mathscr{H}_{2n+1}|| := \sup_{||f||_\infty=1} ||\mathscr{H}_{2n+1}[f]||_\infty = O(1),$$

where $H_n[f]$ and $\mathscr{H}_{2n+1}[f]$ are the Hermite–Fejér interpolation polynomials of a continuous function f on $[-1,1]$ based at the zeros of $E_{n+1}^{(\lambda)}$ and $F_{2n+1}^{(\lambda)}$, respectively.

3. The proofs

First note that the cases $\lambda = 0$ and 1 are considered as

$$E_{n+1}^{(0)}(x) = \frac{2n}{\pi}(T_{n+1}(x) - T_{n-1}(x)),$$
$$E_{n+1}^{(1)}(x) = \frac{2}{\pi}T_{n+1}(x)$$

so that in these cases, the results are well known or can be deduced directly. Therefore, in this paper we consider only the case of $0<\lambda<1$. In the following, we recall several facts from [22]. Let $Q_n^{(\lambda)}$ be the ultraspherical function of the second kind, defined by

$$(1-y^2)^{\lambda-1/2}Q_n^{(\lambda)}(y) = \frac{1}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}\int_{-1}^{1}(1-t^2)^{\lambda-1/2}\frac{P_n^{(\lambda)}(t)}{y-t}\,dt$$

for $y\notin[-1,1]$, $\lambda>-1/2$. For $x\in[-1,1]$, $Q_n^{(\lambda)}(x)$ is defined by Szegő [23, (4.62.9)], or equivalently, by a Cauchy principal value integrals,

$$(1-x^2)^{\lambda-1/2}Q_n^{(\lambda)}(x) = \frac{1}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}PV\int_{-1}^{1}(1-t^2)^{\lambda-1/2}\frac{P_n^{(\lambda)}(t)}{x-t}\,dt.$$

In fact, $f_{v,n}^{(\lambda)}$ defined in (2.3) are the coefficients in the expansion (cf. [22, p. 533]) for $0<\theta<\pi$,

$$\gamma_n^{(\lambda)}\sum_{v=0}^{\infty} f_v e^{-i(n+1+2v)\theta} = \sin^{2\lambda-1}\theta\left(Q_n^{(\lambda)}(\cos\theta) - i\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}P_n^{(\lambda)}(\cos\theta)\right) \quad (3.1)$$

and from the recurrence relation (2.2), we have for $0<\theta<\pi$

$$e^{i(n+1)\theta}\left(\sum_{v=0}^{\infty}\alpha_{v,n}^{(\lambda)}e^{-2iv\theta}\right)\left(\sum_{v=0}^{\infty}f_v e^{-i(n+1+2v)\theta}\right) = 1. \quad (3.2)$$

On the other hand, Szegő [22, p. 509] proved that

$$\alpha_{1,n}^{(\lambda)} < \alpha_{2,n}^{(\lambda)} < \alpha_{3,n}^{(\lambda)} < \cdots < 0 \quad \text{and} \quad 0 \leqslant \sum_{v=0}^{\infty} \alpha_{v,n}^{(\lambda)} < 1. \tag{3.3}$$

For convenience, we consider n even because the case of odd n can be treated analogously and let $m := n/2$.

The following two lemmas play an important role in proving the results.

Lemma 3.1 (Ehrich and Mastroianni [5, Lemma 5.1]). *Let* $0 < \lambda < 1$. *Then for given* $n \in \mathbb{N}$,

$$\sum_{v=0}^{m} \alpha_{v,n}^{(\lambda)} \sim n^{\lambda-1}. \tag{3.4}$$

Lemma 3.2 (Ehrich and Mastroianni [5, Lemma 5.2]). *Let* $0 < \lambda < 1$. *Then for every* $k \leqslant m$,

$$|\alpha_{1,n}^{(\lambda)} + 2\alpha_{2,n}^{(\lambda)} + 3\alpha_{3,n}^{(\lambda)} + \cdots + k\alpha_{k,n}^{(\lambda)}| \leqslant \left(\frac{12}{5}\right)^{\lambda} \frac{\Gamma(2-\lambda)}{\lambda} k^{\lambda}. \tag{3.5}$$

Proof of Proposition 2.2. From (2.1) and (2.7),

$$\begin{aligned}
\frac{d}{d\theta} E_{n+1}^{(\lambda)}(\cos\theta) &= \frac{d}{d\theta}\left(\frac{2}{\gamma_n^{(\lambda)}} \mathrm{Re}\left\{\sum_{v=0}^{m} \alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\}\right) \\
&= -\frac{2}{\gamma_n^{(\lambda)}} \mathrm{Im}\left\{\sum_{v=0}^{m} (n+1-2v)\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\} \\
&= -\frac{2}{\gamma_n^{(\lambda)}}(n+1)\mathrm{Im}\left\{\sum_{v=0}^{m} \alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\} + \frac{4}{\gamma_n^{(\lambda)}} \mathrm{Im}\left\{\sum_{v=1}^{m} v\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\} \\
&= -(n+1)e_n^{(\lambda)}(\theta) + \frac{4}{\gamma_n^{(\lambda)}} \mathrm{Im}\left\{\sum_{v=1}^{m} v\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\}.
\end{aligned}$$

If we let

$$p_1(\theta) := \frac{4}{\gamma_n^{(\lambda)}} \mathrm{Im}\left\{\sum_{v=1}^{m} v\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\},$$

then for $x = \cos\theta$,

$$p_1(\theta)/\sin\theta = \frac{4}{\gamma_n^{(\lambda)}} \sum_{v=1}^{m} v\alpha_{v,n}^{(\lambda)} U_{n-2v}(x),$$

where U_n is the first kind Chebyshev polynomial, so that $p_1(\theta)/\sin\theta \in \mathscr{P}_{n-2}$ for $x = \cos\theta$ and by (2.4) and (3.5),

$$|p_1(\theta)| \leqslant -\frac{4}{\gamma_n^{(\lambda)}} \sum_{v=1}^{m} v\alpha_{v,n}^{(\lambda)} \lesssim n.$$

Similarly, by (2.1) and (2.7),

$$\frac{d}{d\theta} e_n^{(\lambda)}(\theta) = (n+1)E_{n+1}^{(\lambda)}(\cos\theta) - \frac{4}{\gamma_n^{(\lambda)}} \mathrm{Re}\left\{\sum_{v=1}^{m} v\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\}.$$

If we let

$$p_2(\theta) := -\frac{4}{\gamma_n^{(\lambda)}} \mathrm{Re}\left\{\sum_{v=1}^{m} v\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\},$$

then $p_2(\theta) \in \mathscr{P}_{n-1}$ for $x = \cos\theta$ and by (2.4) and (3.5),

$$|p_2(\theta)| \leqslant -\frac{4}{\gamma_n^{(\lambda)}} \sum_{v=1}^{m} v\alpha_{v,n}^{(\lambda)} \lesssim n. \qquad \square$$

Proof of Proposition 2.3. Since

$$\begin{aligned} E_{n+1}^{(\lambda)}{}''(x) &= -\frac{1}{\sin\theta}\frac{d}{d\theta}\left(-\frac{1}{\sin\theta}\frac{d}{d\theta}E_{n+1}^{(\lambda)}(\cos\theta)\right) \\ &= \frac{(E_{n+1}^{(\lambda)}(\cos\theta))_\theta{}''\sin\theta - \cos\theta (E_{n+1}^{(\lambda)}(\cos\theta))_\theta{}'}{\sin^3\theta} \\ &= \frac{1}{1-x^2}\left[(E_{n+1}^{(\lambda)}(\cos\theta))_\theta{}'' + xE_{n+1}^{(\lambda)}{}'(x)\right] \end{aligned}$$

and

$$\begin{aligned} \left(E_{n+1}^{(\lambda)}(\cos\theta)\right)_\theta{}'' &= \frac{2}{\gamma_n^{(\lambda)}} \mathrm{Re}\left\{\sum_{v=0}^{m} -(n+1-2v)^2 \alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\} \\ &= -(n+1)^2 \frac{2}{\gamma_n^{(\lambda)}} \mathrm{Re}\left\{\sum_{v=0}^{m} \alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\} \\ &\quad + (n+1)\frac{8}{\gamma_n^{(\lambda)}} \mathrm{Re}\left\{\sum_{v=1}^{m} v\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\} \\ &\quad - \frac{8}{\gamma_n^{(\lambda)}} \mathrm{Re}\left\{\sum_{v=1}^{m} v^2\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\} \end{aligned}$$

$$= -(n+1)^2 E_{n+1}^{(\lambda)}(\cos\theta) + (n+1)\frac{8}{\gamma_n^{(\lambda)}}\mathrm{Re}\left\{\sum_{v=1}^{m} v\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\}$$
$$- \frac{8}{\gamma_n^{(\lambda)}}\mathrm{Re}\left\{\sum_{v=1}^{m} v^2\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\},$$

if we let

$$I_n(\cos\theta) := (n+1)\frac{8}{\gamma_n^{(\lambda)}}\mathrm{Re}\left\{\sum_{v=1}^{m} v\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\}$$
$$- \frac{8}{\gamma_n^{(\lambda)}}\mathrm{Re}\left\{\sum_{v=1}^{m} v^2\alpha_{v,n}^{(\lambda)} e^{i(n+1-2v)\theta}\right\}$$

then (2.11) is deduced and $I_n(x) \in \mathscr{P}_{n-1}$. Moreover, we have by (2.4) and (3.5),

$$|I_n(\cos\theta)| \leqslant (n+1)\frac{8}{\gamma_n^{(\lambda)}}\sum_{v=1}^{m} -v\alpha_{v,n}^{(\lambda)} - \frac{8}{\gamma_n^{(\lambda)}}\sum_{v=1}^{m} v^2\alpha_{v,n}^{(\lambda)} \lesssim n^2. \qquad \square$$

Proof of Theorem 2.4. First, we consider for $0 \leqslant \theta \leqslant \phi_{n,n}^{(\lambda)}$. Let $s_{n-1,n+1}^{(\lambda)}$ be the greatest zero of $E_{n+1}^{(\lambda)}{}''(x)$ and let $r_{n-1,n+1}^{(\lambda)}$ and $r_{n,n+1}^{(\lambda)}$ be the greatest zeros of $E_{n+1}^{(\lambda)}{}'(x)$ with $r_{n-1,n+1}^{(\lambda)} < r_{n,n+1}^{(\lambda)}$. Then since $r_{n-1,n+1}^{(\lambda)} < s_{n-1,n+1}^{(\lambda)} < r_{n,n+1}^{(\lambda)} < 1$, for $r_{n-1,n+1}^{(\lambda)} \leqslant x \leqslant 1$

$$|E_{n+1}^{(\lambda)}{}'(x)| \leqslant \max\{|E_{n+1}^{(\lambda)}{}'(s_{n-1,n+1}^{(\lambda)})|, |E_{n+1}^{(\lambda)}{}'(1)|\}.$$

By (2.11) and $1 - s_{n-1,n+1}^{(\lambda)} \sim 1/n^2$, we have

$$\begin{aligned}|E_{n+1}^{(\lambda)}{}'(s_{n-1,n+1}^{(\lambda)})| &\leqslant |I_n(s_{n-1,n+1}^{(\lambda)})| + (n+1)^2|E_{n+1}^{(\lambda)}(s_{n-1,n+1}^{(\lambda)})|\\ &\lesssim n^2 + (n+1)^2(n^{1-\lambda}\varphi^{1-\lambda}(s_{n-1,n+1}^{(\lambda)}) + 1)\\ &\lesssim n^2\end{aligned}$$

and from [5, Lemma 5.3], we have $|E_{n+1}^{(\lambda)}{}'(1)| \lesssim n^2$. Therefore, we have for $r_{n-1,n+1}^{(\lambda)} \leqslant x \leqslant 1$

$$|E_{n+1}^{(\lambda)}{}'(x)| \lesssim n^2.$$

Then since $r_{n-1,n+1}^{(\lambda)} < \xi_{n,n+1}^{(\lambda)} < x_{n,n}^{(\lambda)} < \xi_{n,n+1}^{(\lambda)} < 1$, for $0 \leqslant \theta \leqslant \phi_{n,n}^{(\lambda)}$ by (2.5)

$$\left|\frac{d}{d\theta}E_{n+1}^{(\lambda)}(\cos\theta)\right| \lesssim n^2\sin\theta \lesssim n$$

so that by (2.8) and (2.10) we have for $\theta \in [0, \phi_{n,n}^{(\lambda)}] \cup [\phi_{1,n}^{(\lambda)}, \pi]$,

$$|e_n^{(\lambda)}(\theta)| \leqslant \frac{1}{n+1}\left|\frac{d}{d\theta}E_{n+1}^{(\lambda)}(\cos\theta)\right| + \frac{|p_1(\theta)|}{n+1} \lesssim 1. \tag{3.6}$$

For $\phi_{n,n}^{(\lambda)} \leqslant \theta \leqslant \phi_{1,n}^{(\lambda)}$, it follows the proof of Theorem 2.1 in [6]. Let

$$D_n(\theta,\lambda) := \sin^{2\lambda}\theta \left\{ \left[\frac{2}{\pi} \frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} Q_n^{(\lambda)}(\cos\theta) \right]^2 + [P_n^{(\lambda)}(\cos\theta)]^2 \right\}, \tag{3.7}$$

then by Durand's result [2] (cf. [6, (15)]), for $\phi_{n,n}^{(\lambda)} \leqslant \theta \leqslant \phi_{1,n}^{(\lambda)}$

$$\frac{4}{\pi^2} \frac{\Gamma^2(\lambda+1/2)}{\Gamma^2(2\lambda)} \sin^{2\lambda} \phi_{n,n}^{(\lambda)} [Q_n^{(\lambda)}(\cos \phi_{n,n}^{(\lambda)})]^2 < D_n(\theta,\lambda) < \left(\frac{\Gamma(n/2+\lambda)}{\Gamma(\lambda)\Gamma(n/2+1)} \right)^2,$$

where $\cos\phi_{n,n}^{(\lambda)} = x_{n,n}^{(\lambda)}$ is the largest zero of $P_n^{(\lambda)}$. If we let $a_{v,n}^{(\lambda)}$ the Gaussian quadrature weights in the ultraspherical case(cf. e.g. [11, p. 94]), then we have from the representation of $a_{v,n}^{(\lambda)}$ and a lower bound of $a_{v,n}^{(\lambda)}$ in [7]

$$\begin{aligned} -\frac{Q_n^{(\lambda)}(\cos\phi_{n,n}^{(\lambda)}) \sin^{2\lambda-1}\phi_{n,n}^{(\lambda)}}{P_n^{(\lambda)\prime}(\cos\phi_{n,n}^{(\lambda)})} &= \frac{1}{2} \frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)} a_{n,n}^{(\lambda)} \\ &\geqslant \frac{1}{2} \frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)} \frac{3\lambda^2+\lambda+1/4}{2(\lambda+1/2)^2} \frac{\pi}{n+\lambda} \sin^{2\lambda}\phi_{n,n}^{(\lambda)}. \end{aligned}$$

So, we have by (8.9.7) in [23]

$$|Q_n^{(\lambda)}(\cos\phi_{n,n}^{(\lambda)})| \gtrsim \frac{\sin\phi_{n,n}^{(\lambda)}}{n} |P_n^{(\lambda)\prime}(\cos\phi_{n,n}^{(\lambda)})| \sim n^{2\lambda-1}.$$

Therefore, we have for $\phi_{n,n}^{(\lambda)} \leqslant \theta \leqslant \phi_{1,n}^{(\lambda)}$

$$D_n(\theta,\lambda) \sim n^{2\lambda-2}. \tag{3.8}$$

Now, we have from (2.7),

$$\begin{aligned} &|e_n^{(\lambda)}(\theta)| \\ &\quad \leqslant \frac{2}{\gamma_n^{(\lambda)}} \left(\left| \sum_{v=0}^{\infty} \alpha_{v,n}^{(\lambda)} \sin(n+1-2v)\theta \right| + \left| \sum_{v=m+1}^{\infty} \alpha_{v,n}^{(\lambda)} \sin(n+1-2v)\theta \right| \right). \end{aligned}$$

Then since by (3.1), (3.2), (3.8), and (7.33.5) in [23], for $\phi_{n,n}^{(\lambda)} \leqslant \theta \leqslant \phi_{1,n}^{(\lambda)}$

$$\begin{aligned} \frac{2}{\gamma_n^{(\lambda)}} \left| \sum_{v=0}^{\infty} \alpha_{v,n}^{(\lambda)} \sin(n+1-2v)\theta \right| &= \frac{4}{\pi} \frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} \left| \frac{\sin\theta P_n^{(\lambda)}(\cos\theta)}{D_n(\theta,\lambda)} \right| \\ &\lesssim n^{2-2\lambda} n^{\lambda-1} \varphi^{-\lambda+1}(x) = n^{1-\lambda} \varphi^{1-\lambda}(x) \end{aligned}$$

and by (3.3) and (3.4),

$$\left|\sum_{v=m+1}^{\infty}\alpha_{v,n}^{(\lambda)}\sin(n+1-2v)\theta\right|\leqslant -\sum_{v=m+1}^{\infty}\alpha_{v,n}^{(\lambda)}\leqslant\sum_{v=0}^{m}\alpha_{v,n}^{(\lambda)}=O(n^{\lambda-1}), \tag{3.9}$$

we have for $\phi_{n,n}^{(\lambda)}\leqslant\theta\leqslant\phi_{1,n}^{(\lambda)}$,

$$|e_n^{(\lambda)}(\theta)|\lesssim n^{1-\lambda}\varphi^{1-\lambda}(x)+1\lesssim n^{1-\lambda}\varphi^{1-\lambda}(x). \tag{3.10}$$

Therefore, from (3.6) and (3.10) we have for $0\leqslant\theta\leqslant\pi$,

$$|e_n^{(\lambda)}(\theta)|\lesssim n^{1-\lambda}\varphi^{1-\lambda}(x)+1. \qquad \square$$

Proof of Theorem 2.5. (a) Since for $0<\theta<\pi$ by (2.8), (2.10), and (2.13),

$$\begin{aligned}\left|\frac{d}{d\theta}E_{n+1}^{(\lambda)}(\cos\theta)\right| &\lesssim (n+1)|e_n^{(\lambda)}(\theta)|+|p_1(\theta)|\\ &\lesssim n^{2-\lambda}\varphi^{1-\lambda}(x)+n,\end{aligned}$$

we have for $x\in[\xi_{1,n+1}^{(\lambda)},\xi_{n+1,n+1}^{(\lambda)}]$,

$$|E_{n+1}^{(\lambda)}{}'(x)|\lesssim n^{2-\lambda}\varphi^{-\lambda}(x).$$

Therefore, we have (2.14). On the other hand, it was proved that $|E_{n+1}^{(\lambda)}{}'(1)|\lesssim n^2$ in [5, Lemma 5.3] and we will estimate the lower bound of $|E_{n+1}^{(\lambda)}{}'(1)|$. Since

$$\frac{E_{n+1}^{(\lambda)}{}'(x)}{E_{n+1}^{(\lambda)}(x)}=\sum_{i=1}^{n+1}\frac{1}{x-\xi_{i,n+1}^{(\lambda)}},$$

we have by (2.5)

$$\frac{E_{n+1}^{(\lambda)}{}'(1)}{E_{n+1}^{(\lambda)}(1)}=\sum_{i=1}^{n+1}\frac{1}{1-\xi_{i,n+1}^{(\lambda)}}\geqslant\frac{1}{1-\xi_{n+1,n+1}^{(\lambda)}}\sim n^2.$$

So, we have by Proposition 2.1,

$$E_{n+1}^{(\lambda)}{}'(1)\gtrsim n^2E_{n+1}^{(\lambda)}(1)\sim n^2.$$

Therefore, we have $E_{n+1}^{(\lambda)}{}'(1)\sim n^2$. Since $E_{n+1}^{(\lambda)}{}'(x)$ is increasing on $[\xi_{n+1,n+1}^{(\lambda)},1]$, we have for $x\in[-1,\xi_{1,n+1}^{(\lambda)}]\cup[\xi_{n+1,n+1}^{(\lambda)},1]$ by (2.5) and (2.24)

$$n^2\sim|E_{n+1}^{(\lambda)}{}'(\xi_{n+1,n+1}^{(\lambda)})|\leqslant|E_{n+1}^{(\lambda)}{}'(x)|\leqslant|E_{n+1}^{(\lambda)}{}'(1)|\sim n^2.$$

Therefore, we have (2.15).

(b) By (2.6), (2.11), (2.12), and (2.14), we have for $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,

$$
\begin{aligned}
|E_{n+1}^{(\lambda)}{}''(x)| &\lesssim \frac{1}{1-x^2}(n^2|E_{n+1}^{(\lambda)}(x)| + |x||E_{n+1}^{(\lambda)}{}'(x)| + n^2) \\
&\lesssim \varphi^{-2}(x)(n^{3-\lambda}\varphi^{1-\lambda}(x) + n^{2-\lambda}\varphi^{-\lambda}(x) + n^2) \\
&\lesssim n^{3-\lambda}\varphi^{-1-\lambda}(x).
\end{aligned}
$$

Therefore, (2.16) is proved. On the other hand, since by (2.5)

$$
\begin{aligned}
\frac{E_{n+1}^{(\lambda)}{}''(\xi_{n+1,n+1}^{(\lambda)})}{E_{n+1}^{(\lambda)}{}'(\xi_{n+1,n+1}^{(\lambda)})} &= \sum_{i=1}^{n} \frac{1}{\xi_{n+1,n+1}^{(\lambda)} - r_{i,n+1}^{(\lambda)}} \geqslant \frac{1}{\xi_{n+1,n+1}^{(\lambda)} - r_{n,n+1}^{(\lambda)}} \\
&\geqslant \frac{1}{\xi_{n+1,n+1}^{(\lambda)} - \xi_{n,n+1}^{(\lambda)}} \gtrsim n^2,
\end{aligned}
$$

we have by (2.24)

$$
E_{n+1}^{(\lambda)}{}''(\xi_{n+1,n+1}^{(\lambda)}) \gtrsim n^2 E_{n+1}^{(\lambda)}{}'(\xi_{n+1,n+1}^{(\lambda)}) \sim n^4,
$$

where $r_{i,n+1}^{(\lambda)}$ are the zeros of $E_{n+1}^{(\lambda)}{}'(x)$ so that we have from (2.16)

$$
|E_{n+1}^{(\lambda)}{}''(\xi_{n+1,n+1}^{(\lambda)})| \sim n^4.
$$

Since it was proved that $E_{n+1}^{(\lambda)}{}''(1) \lesssim n^4$ in [5, Lemma 5.4], we estimate the lower bound of $E_{n+1}^{(\lambda)}{}''(1)$. Since by (2.5)

$$
\frac{E_{n+1}^{(\lambda)}{}''(1)}{E_{n+1}^{(\lambda)}{}'(1)} \geqslant \frac{1}{1 - r_{n,n+1}^{(\lambda)}} \geqslant \frac{1}{1 - \xi_{n,n+1}^{(\lambda)}} \gtrsim n^2,
$$

we have

$$
E_{n+1}^{(\lambda)}{}''(1) \gtrsim n^2 E_{n+1}^{(\lambda)}{}'(1) \sim n^4
$$

so that we have $E_{n+1}^{(\lambda)}{}''(1) \sim n^4$. Here, since $E_{n+1}^{(\lambda)}{}''(x)$ is increasing on $[\xi_{n+1,n+1}^{(\lambda)}, 1]$, we have for $x \in [-1, \xi_{1,n+1}^{(\lambda)}] \cup [\xi_{n+1,n+1}^{(\lambda)}, 1]$,

$$
n^4 \sim |E_{n+1}^{(\lambda)}{}''(\xi_{n+1,n+1}^{(\lambda)})| \leqslant |E_{n+1}^{(\lambda)}{}''(x)| \leqslant |E_{n+1}^{(\lambda)}{}''(1)| \sim n^4.
$$

Therefore, (2.17) is proved. □

Proof of Corollary 2.6. By (2.11) and (4.2.1) in [23],

$$\begin{aligned}
&(1-x^2)F_{2n+1}^{(\lambda)}{}''(x)\\
&= (1-x^2)E_{n+1}^{(\lambda)}{}''(x)P_n^{(\lambda)}(x) + 2(1-x^2)E_{n+1}^{(\lambda)}{}'(x)P_n^{(\lambda)\prime}(x) + E_{n+1}^{(\lambda)}(1-x^2)P_n^{(\lambda)\prime\prime}(x)\\
&= (-(n+1)^2E_{n+1}^{(\lambda)}(x) + xE_{n+1}^{(\lambda)}{}'(x) + I_n(x))P_n^{(\lambda)}(x) + 2(1-x^2)E_{n+1}^{(\lambda)}{}'(x)P_n^{(\lambda)\prime}(x)\\
&\quad + ((2\lambda+1)xP_n^{(\lambda)\prime}(x) - n(n+2\lambda)P_n^{(\lambda)}(x))E_{n+1}^{(\lambda)}(x)\\
&= (-(n+1)^2 - n(n+2\lambda))E_{n+1}^{(\lambda)}(x)P_n^{(\lambda)}(x) + x(E_{n+1}^{(\lambda)}{}'(x)P_n^{(\lambda)}(x) + E_{n+1}^{(\lambda)}(x)P_n^{(\lambda)\prime}(x))\\
&\quad + 2\lambda xP_n^{(\lambda)\prime}(x)E_{n+1}^{(\lambda)}(x) + 2(1-x^2)E_{n+1}^{(\lambda)}{}'(x)P_n^{(\lambda)\prime}(x) + I_n(x)P_n^{(\lambda)}(x)\\
&= -\big(2n^2+2(1+\lambda)n+1\big)F_{2n+1}^{(\lambda)}(x) + xF_{2n+1}^{(\lambda)}{}'(x) + J_n(x),
\end{aligned}$$

where

$$J_n(x) := 2\lambda xE_{n+1}^{(\lambda)}(x)P_n^{(\lambda)\prime}(x) + 2(1-x^2)E_{n+1}^{(\lambda)}{}'(x)P_n^{(\lambda)\prime}(x) + I_n(x)P_n^{(\lambda)}(x). \quad (3.11)$$

Then for $x\in[\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$ by (2.6), (2.12), (2.14), (4.7.27) and (7.33.5) in [23],

$$\begin{aligned}
|J_n(x)| &\lesssim |E_{n+1}^{(\lambda)}(x)||P_n^{(\lambda)\prime}(x)| + \varphi^2(x)|E_{n+1}^{(\lambda)}{}'(x)||P_n^{(\lambda)\prime}(x)| + |I_n(x)||P_n^{(\lambda)}(x)|\\
&\lesssim (n^{1-\lambda}\varphi^{1-\lambda}(x) + n^{2-\lambda}\varphi^{2-\lambda}(x))|P_{n-1}^{\lambda+1}(x)| + n^{\lambda+1}\varphi^{-\lambda}(x)\\
&\lesssim n^2\varphi^{1-2\lambda}(x) + n^{\lambda+1}\varphi^{-\lambda}(x)\\
&\lesssim n^2\varphi^{1-2\lambda}(x)
\end{aligned}$$

and for $x\in[-1, \xi_{1,n+1}^{(\lambda)}]\cup[\xi_{n+1,n+1}^{(\lambda)}, 1]$ by (2.6), (2.12), and (2.15),

$$\begin{aligned}
|J_n(x)| &\lesssim |P_n^{(\lambda)\prime}(x)| + |P_n^{(\lambda)\prime}(x)| + n^2|P_n^{(\lambda)}(x)|\\
&\lesssim n^{1+2\lambda}. \qquad \square
\end{aligned}$$

Proof of Corollary 2.7. For $x\in[\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$, by (2.6), (2.12), (2.14), (4.7.27) and (7.33.5) in [23],

$$\begin{aligned}
|F_{2n+1}^{(\lambda)}{}'(x)| &= |E_{n+1}^{(\lambda)}{}'(x)P_n^{(\lambda)}(x) + E_{n+1}^{(\lambda)}(x)P_n^{(\lambda)\prime}(x)|\\
&\lesssim n^{2-\lambda}\varphi^{-\lambda}(x)n^{\lambda-1}\varphi^{-\lambda}(x) + n^{1-\lambda}\varphi^{1-\lambda}(x)n^{\lambda}\varphi^{-\lambda-1}(x)\\
&\lesssim n\varphi^{-2\lambda}(x)
\end{aligned}$$

and by (2.18) and (2.19)

$$\begin{aligned}
|F_{2n+1}^{(\lambda)}{}''(x)| &\lesssim \varphi^{-2}(x)(n^2|F_{2n+1}^{(\lambda)}(x)| + |F_{n+1}^{(\lambda)}{}'(x)| + n^2\varphi^{1-2\lambda}(x))\\
&\lesssim \varphi^{-2}(x)(n^2n^{1-\lambda}\varphi^{1-\lambda}(x)n^{\lambda-1}\varphi^{-\lambda}(x) + n\varphi^{-2\lambda}(x) + n^2\varphi^{1-2\lambda}(x))\\
&\lesssim n^2\varphi^{-1-2\lambda}(x).
\end{aligned}$$

Therefore, (2.20) and (2.22) are proved. Now, we prove (2.21) and (2.23). From Proposition 2.1, the normalization of $P_n^{(\lambda)}$, (2.15), (2.17), and (4.7.27) in [23], we have

$$|F_{2n+1}^{(\lambda)}(1)| = |E_{n+1}^{(\lambda)}(1)P_n^{(\lambda)}(1)| \sim n^{2\lambda-1},$$

$$|F_{2n+1}^{(\lambda)}{}'(1)| = |E_{n+1}^{(\lambda)}{}'(1)P_n^{(\lambda)}(1) + E_{n+1}^{(\lambda)}(1)P_n^{(\lambda)\prime}(1)| \lesssim n^2 n^{2\lambda-1} + n^{2\lambda+1} \lesssim n^{1+2\lambda},$$

and

$$\begin{aligned}|F_{2n+1}^{(\lambda)}{}''(1)| &= |E_{n+1}^{(\lambda)}{}''(1)P_n^{(\lambda)}(1) + 2E_{n+1}^{(\lambda)}{}'(1)P_n^{(\lambda)\prime}(1) + E_{n+1}^{(\lambda)}(1)P_n^{(\lambda)\prime\prime}(1)|\\ &\lesssim n^4 n^{2\lambda-1} + n^2 n^{2\lambda+1} + n^{2\lambda+3} \lesssim n^{2\lambda+3}.\end{aligned}$$

The rest can be proved by the same methods as the proof of Theorem 2.5. Therefore, (2.21) and (2.23) are proved. □

To prove Theorem 2.9, we need the following lemmas.

Lemma 3.3. *Let $\lambda \in (0,1)$. Then for $v = 1, 2, \ldots, n$*

$$|e_n^{(\lambda)}(\phi_{v,n}^{(\lambda)})| \lesssim 1 \tag{3.12}$$

and

$$\left|\frac{d}{d\theta} E_{n+1}^{(\lambda)}(\cos\theta)\right|_{\theta=\phi_{v,n}^{(\lambda)}} \lesssim n. \tag{3.13}$$

Lemma 3.4. *Let $\lambda \in (0,1)$. Then for $\mu = 1, 2, \ldots, n+1$*

$$\left|\frac{d}{d\theta} e_n^{(\lambda)}(\theta)\right|_{\theta=\theta_{\mu,n+1}^{(\lambda)}} \lesssim n \tag{3.14}$$

and

$$\left|\frac{d}{d\theta} P_n^{(\lambda)}(\cos\theta)\right|_{\theta=\theta_{\mu,n+1}^{(\lambda)}} \lesssim n^{2\lambda-1}\varphi^{-1}(\cos\theta_{\mu,n+1}^{(\lambda)}). \tag{3.15}$$

Remark 3.5. For $|E_{n+1}^{(\lambda)}(x_{v,n}^{(\lambda)})|$ and $|P_n^{(\lambda)}(\xi_{\mu,n+1}^{(\lambda)})|$, we can deduce easily from (2.25) that

$$|E_{n+1}^{(\lambda)}(x_{v,n}^{(\lambda)})| \sim n^{1-\lambda}\varphi^{1-\lambda}(x_{v,n}^{(\lambda)})$$

and

$$|P_n^{(\lambda)}(\xi_{\mu,n+1}^{(\lambda)})| \sim n^{\lambda-1}\varphi^{-\lambda}(\xi_{\mu,n+1}^{(\lambda)}).$$

Proof of Lemma 3.3. Since by (2.7) for $0<\theta<\pi$,

$$e_n^{(\lambda)}(\cos\theta) = \frac{2}{\gamma_n^{(\lambda)}}\left(\mathrm{Im}\sum_{v=0}^{\infty}\alpha_{v,n}^{(\lambda)}e^{i(n+1-2v)\theta} - \mathrm{Im}\sum_{v=m+1}^{\infty}\alpha_{v,n}^{(\lambda)}e^{i(n+1-2v)\theta}\right),$$

we have by (3.1) and (3.2)

$$\begin{aligned}|e_n^{(\lambda)}(\theta)| \leqslant \sin^{1-2\lambda}\theta\,\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}|P_n^{(\lambda)}(\cos\theta)| \\ \times\left([Q_n^{(\lambda)}(\cos\theta)]^2 + \left[\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}P_n^{(\lambda)}(\cos\theta)\right]^2\right)^{-1} \\ + \frac{2}{\gamma_n^{(\lambda)}}\left|\sum_{v=m+1}^{\infty}\alpha_{v,n}^{(\lambda)}\sin(n+1-2v)\theta\right|.\end{aligned}$$

Then by (2.4) and (3.9) for $v=1,2,\dots,n$,

$$|e_n^{(\lambda)}(\phi_{v,n}^{(\lambda)})| \leqslant \frac{2}{\gamma_n^{(\lambda)}}\left|\sum_{v=m+1}^{\infty}\alpha_{v,n}^{(\lambda)}\sin(n+1-2v)\phi_{v,n}^{(\lambda)}\right| = O(1)$$

so that by (2.8) and (2.10), we also have

$$\begin{aligned}\left|\frac{d}{d\theta}E_{n+1}^{(\lambda)}(\cos\theta)\right|_{\theta=\phi_{v,n}^{(\lambda)}} &\leqslant (n+1)|e_n^{(\lambda)}(\phi_{v,n}^{(\lambda)})| + |p_2(\phi_{v,n}^{(\lambda)})| \\ &\lesssim n.\end{aligned}$$

Therefore, we have (3.12) and (3.13). □

Proof of Lemma 3.4. From (2.9) and (2.10), we have easily (3.14). Now, we prove (3.15). Since for $\mu=1$ and $n+1$ by (4.7.27), (7.33.5) in [23] and (2.5)

$$\left|\frac{d}{d\theta}P_n^{(\lambda)}(\cos\theta)\right|_{\theta=\theta_{\mu,n+1}^{(\lambda)}} \lesssim n^{\lambda}\varphi^{-\lambda}(\cos\theta_{\mu,n+1}^{(\lambda)}) \sim n^{2\lambda-1}\varphi^{-1}(\cos\theta_{\mu,n+1}^{(\lambda)}) \sim n^{2\lambda},$$

we have (3.15) for $\theta_{1,n+1}^{(\lambda)}$ and $\theta_{n+1,n+1}^{(\lambda)}$ so that we will prove (3.15) only for $\mu = 2,\dots,n$. Let

$$\begin{aligned}&e^{i(n+1)\theta}\sum_{v=0}^{\infty}\alpha_{v,n}^{(\lambda)}e^{-2iv\theta} \\ &\quad =: \frac{\gamma_n^{(\lambda)}}{2}(E_{n+1}^{(\lambda)}(\cos\theta)+h_E(\theta)) + i\frac{\gamma_n^{(\lambda)}}{2}(e_n^{(\lambda)}(\theta)+h_e(\theta)),\end{aligned} \tag{3.16}$$

where

$$h_E(\theta) + ih_e(\theta) := \frac{2}{\gamma_n^{(\lambda)}}e^{i(n+1)\theta}\sum_{v=m+1}^{\infty}\alpha_{v,n}^{(\lambda)}e^{-2iv\theta}.$$

Then since by the relation (3.2)

$$\begin{aligned}\sum_{v=0}^{\infty} f_v e^{-i(n+1+2v)\theta} &= \left(e^{i(n+1)\theta}\sum_{v=0}^{\infty}\alpha_{v,n}^{(\lambda)}e^{-2iv\theta}\right)^{-1}\\ &=\frac{2}{\gamma_n^{(\lambda)}}([E_{n+1}^{(\lambda)}(\cos\theta)+h_E(\theta)]+i[e_n^{(\lambda)}(\theta)+h_e(\theta)])^{-1}\\ &=\frac{2}{\gamma_n^{(\lambda)}}\frac{[E_{n+1}^{(\lambda)}(\cos\theta)+h_E(\theta)]-i[e_n^{(\lambda)}(\theta)+h_e(\theta)]}{[E_{n+1}^{(\lambda)}(\cos\theta)+h_E(\theta)]^2+[e_n^{(\lambda)}(\theta)+h_e(\theta)]^2}\end{aligned}$$

and by (3.1)

$$\sum_{v=0}^{\infty} f_v e^{-i(n+1+2v)\theta}=\frac{1}{\gamma_n^{(\lambda)}}\sin^{2\lambda-1}\theta\left(Q_n^{(\lambda)}(\cos\theta)-i\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}P_n^{(\lambda)}(\cos\theta)\right),$$

by comparing the imaginary parts, we have

$$\begin{aligned}&\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}\frac{1}{\gamma_n^{(\lambda)}}\sin^{2\lambda-1}\theta P_n^{(\lambda)}(\cos\theta)\\ &\quad=\frac{2}{\gamma_n^{(\lambda)}}\frac{e_n^{(\lambda)}(\theta)+h_e(\theta)}{[E_{n+1}^{(\lambda)}(\cos\theta)+h_E(\theta)]^2+[e_n^{(\lambda)}(\theta)+h_e(\theta)]^2}.\end{aligned}$$

If we differentiate the above equation for θ, then

$$\begin{aligned}&\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}\frac{1}{\gamma_n^{(\lambda)}}(2\lambda-1)\sin^{2\lambda-2}\theta\cos\theta P_n^{(\lambda)}(\cos\theta)\\ &\quad+\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}\frac{1}{\gamma_n^{(\lambda)}}\sin^{2\lambda-1}\theta\frac{d}{d\theta}P_n^{(\lambda)}(\cos\theta)\\ &\quad=\frac{2}{\gamma_n^{(\lambda)}}[(e_n^{(\lambda)}+h_e)'[(E_{n+1}^{(\lambda)}+h_E)^2+(e_n^{(\lambda)}+h_e)^2]\\ &\qquad-(e_n^{(\lambda)}+h_e)[2(E_{n+1}^{(\lambda)}{}'+h_E{}')(E_{n+1}^{(\lambda)}+h_E)+2(e_n^{(\lambda)\prime}+h_e{}')(e_n^{(\lambda)}+h_e)]]\\ &\qquad\times([E_{n+1}^{(\lambda)}+h_E]^2+[e_n^{(\lambda)}+h_e]^2)^{-2}.\end{aligned}$$

First, we estimate $h_E(\theta)$, $h_e(\theta)$, $h_E{}'(\theta)$, and $h_e{}'(\theta)$. By (2.4) and (3.9)

$$|h_e(\theta)|=\frac{2}{\gamma_n^{(\lambda)}}\left|\sum_{v=m+1}^{\infty}\alpha_{v,n}^{(\lambda)}\sin(n+1-2v)\theta\right|=O(1) \tag{3.17}$$

and similarly

$$|h_E(\theta)|=\frac{2}{\gamma_n^{(\lambda)}}\left|\sum_{v=m+1}^{\infty}\alpha_{v,n}^{(\lambda)}\cos(n+1-2v)\theta\right|=O(1). \tag{3.18}$$

Now, we estimate $|h_E{}'(\theta)|$ and $|h_e{}'(\theta)|$ using the ideas of Ehrich's paper [3]. For $|h_E{}'(\theta)|$, we obtain

$$\begin{aligned}|h_E{}'(\theta)| &= \frac{2}{\gamma_n^{(\lambda)}}\left|\sum_{v=m+1}^{\infty} -\alpha_{v,n}^{(\lambda)}(n+1-2v)\sin(n+1-2v)\theta\right| \\ &= \frac{2}{\gamma_n^{(\lambda)}}\left|\sum_{v=1}^{\infty} -\alpha_{v+m,n}^{(\lambda)}(2v-1)\sin(2v-1)\theta\right| \\ &\leqslant \frac{4}{\gamma_n^{(\lambda)}}\left|\sum_{v=1}^{\infty} v\alpha_{m+v,n}^{(\lambda)}\sin 2v\theta\cos\theta\right| + \frac{4}{\gamma_n^{(\lambda)}}\left|\sum_{v=1}^{\infty} v\alpha_{m+v,n}^{(\lambda)}\cos 2v\theta\sin\theta\right| \\ &\quad + \frac{2}{\gamma_n^{(\lambda)}}\left|\sum_{v=1}^{\infty}\alpha_{m+v,n}^{(\lambda)}\sin(2v-1)\theta\right|.\end{aligned}$$

Then we have by partial summation,

$$\begin{aligned}&\sum_{v=1}^{\infty} v\alpha_{m+v,n}^{(\lambda)}\sin 2v\theta \\ &\quad = \lim_{K\to\infty}\left[\sum_{v=1}^{K-1}(v\alpha_{m+v,n}^{(\lambda)}-(v+1)\alpha_{m+v+1,n}^{(\lambda)})\sum_{\mu=1}^{v}\sin 2\mu\theta + K\alpha_{m+K,n}^{(\lambda)}\sum_{\mu=1}^{K}\sin 2\mu\theta\right].\end{aligned}$$

Here, we know that

$$\left|\sum_{\mu=1}^{v}\sin 2\mu\theta\right| = \left|\frac{\cos\theta-\cos(2v+1)\theta}{2\sin\theta}\right| \leqslant \frac{1}{\sin\theta}$$

and we have by (3.3) and (3.9)

$$|K\alpha_{m+K,n}^{(\lambda)}| \leqslant \sum_{v=1}^{K} -\alpha_{m+v,n}^{(\lambda)} \leqslant \sum_{v=1}^{\infty} -\alpha_{m+v,n}^{(\lambda)} = O(n^{\lambda-1})$$

and

$$\begin{aligned}&\sum_{v=1}^{K-1}|v\alpha_{m+v,n}^{(\lambda)}-(v+1)\alpha_{m+v+1,n}^{(\lambda)}| \\ &\quad \leqslant \sum_{v=1}^{K-1} v|\alpha_{m+v,n}^{(\lambda)}-\alpha_{m+v+1,n}^{(\lambda)}| + \sum_{v=1}^{K-1}|\alpha_{m+v+1,n}^{(\lambda)}| \\ &\quad = \sum_{v=1}^{K-1} -\alpha_{m+v,n}^{(\lambda)} + (K-1)\alpha_{m+K,n}^{(\lambda)} + \sum_{v=1}^{K-1} -\alpha_{m+v+1,n}^{(\lambda)} \\ &\quad = O(n^{\lambda-1}).\end{aligned}$$

Therefore, we have for $0<\theta<\pi$,

$$\left|\sum_{v=1}^{\infty} v\alpha_{m+v,n}^{(\lambda)} \sin 2v\theta\right| \lesssim \frac{n^{\lambda-1}}{\sin\theta}.$$

Similarly, since

$$\left|\sum_{\mu=1}^{v} \cos 2\mu\theta\right| = \left|\frac{\sin(2v+1)\theta - \sin\theta}{2\sin\theta}\right| \leqslant \frac{1}{\sin\theta},$$

we have for $0<\theta<\pi$,

$$\left|\sum_{v=1}^{\infty} v\alpha_{m+v,n}^{(\lambda)} \cos 2v\theta\right| \lesssim \frac{n^{\lambda-1}}{\sin\theta}$$

and by (3.9)

$$\left|\sum_{v=1}^{\infty} \alpha_{m+v,n}^{(\lambda)} \sin(2v-1)\theta\right| \lesssim n^{\lambda-1}.$$

Therefore, by (2.4) we have for $0<\theta<\pi$,

$$|h_E{}'(\theta)| \lesssim \frac{1}{\sin\theta}. \tag{3.19}$$

By the same reason as the above, we have for $0<\theta<\pi$,

$$\begin{aligned} |h_e{}'(\cos\theta)| &= \frac{2}{\gamma_n^{(\lambda)}}\left|\sum_{v=m+1}^{\infty} \alpha_{v,n}^{(\lambda)}(n+1-2v)\cos(n+1-2v)\theta\right| \\ &\lesssim \frac{1}{\sin\theta}. \end{aligned} \tag{3.20}$$

Next, we obtain by (3.2), (3.7), and (3.16),

$$\begin{aligned} &([E_{n+1}^{(\lambda)}(\cos\theta) + h_E(\theta)]^2 + [e_n^{(\lambda)}(\theta) + h_e(\theta)]^2)^{1/2} \\ &\quad = \frac{2}{\gamma_n^{(\lambda)}}\left|\sum_{v=0}^{\infty} f_v e^{-i(n+1+2v)\theta}\right|^{-1} \\ &\quad = 2\sin^{1-2\lambda}\theta\left([Q_n^{(\lambda)}(\cos\theta)]^2 + \left[\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)} P_n^{(\lambda)}(\cos\theta)\right]^2\right)^{-1/2} \\ &\quad = 2\sin^{1-\lambda}\theta D_n^{-1/2}(\theta)\frac{2}{\pi}\frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} \end{aligned}$$

so that we have by (3.8) for $\phi_{n,n}^{(\lambda)} \leqslant \theta \leqslant \phi_{1,n}^{(\lambda)}$

$$([E_{n+1}^{(\lambda)}(\cos\theta) + h_E(\cos\theta)]^2 + [e_n^{(\lambda)}(\cos\theta) + h_e(\cos\theta)]^2)^{1/2} \sim n^{1-\lambda}\varphi^{1-\lambda}(\cos\theta).$$

Then by (3.14), (3.20), (3.18), (3.12), and (3.17), for $\mu = 2, 3, \ldots, n$,

$$\begin{aligned} &|(e_n^{(\lambda)} + h_e)'[(E_{n+1}^{(\lambda)} + h_E)^2 + (e_n^{(\lambda)} + h_e)^2](\theta_{\mu,n+1}^{(\lambda)})| \\ &\quad \lesssim \left(O(n) + O(1)\frac{1}{\sin\theta_{\mu,n+1}^{(\lambda)}}\right)[O(1) + (O(n^{1-\lambda})\sin^{1-\lambda}\theta_{\mu,n+1}^{(\lambda)})^2] \\ &\quad \lesssim O(n^{3-2\lambda})\sin^{2-2\lambda}\theta_{\mu,n+1}^{(\lambda)}, \end{aligned} \tag{3.21}$$

by (2.13), (3.17), (2.24), (3.19), and (3.18),

$$\begin{aligned} &|(e_n^{(\lambda)} + h_e)(E_{n+1}^{(\lambda)}{}' + h_E{}')(E_{n+1}^{(\lambda)} + h_E)(\theta_{\mu,n+1}^{(\lambda)})| \\ &\quad \lesssim O(n^{1-\lambda})\sin^{1-\lambda}\theta_{\mu,n+1}^{(\lambda)}\left(O(n^{2-\lambda})\sin^{1-\lambda}\theta_{\mu,n+1}^{(\lambda)} + O(1)\frac{1}{\sin\theta_{\mu,n+1}^{(\lambda)}}\right) \\ &\quad \lesssim O(n^{3-2\lambda})\sin^{2-2\lambda}\theta_{\mu,n+1}^{(\lambda)}, \end{aligned} \tag{3.22}$$

and by (2.13), (3.17), (3.14), and (3.20),

$$\begin{aligned} &|(e_n^{(\lambda)} + h_e)(e_n^{(\lambda)\prime} + h_e{}')(e_n^{(\lambda)} + h_e)(\theta_{\mu,n+1}^{(\lambda)})| \\ &\quad \lesssim O(n^{1-\lambda})\sin^{1-\lambda}\theta_{\mu,n+1}^{(\lambda)}\left(O(n) + O(1)\frac{1}{\sin\theta_{\mu,n+1}^{(\lambda)}}\right)O(n^{1-\lambda})\sin^{1-\lambda}\theta_{\mu,n+1}^{(\lambda)} \\ &\quad \lesssim O(n^{3-2\lambda})\sin^{2-2\lambda}\theta_{\mu,n+1}^{(\lambda)}. \end{aligned} \tag{3.23}$$

Therefore, we have from (3.21)–(3.23),

$$\begin{aligned} &|(e_n^{(\lambda)} + h_e)'[(E_{n+1}^{(\lambda)} + h_E)^2 + (e_n^{(\lambda)} + h_e)^2] \\ &\quad - (e_n^{(\lambda)} + h_e)[2(E_{n+1}^{(\lambda)}{}' + h_E{}')(E_{n+1}^{(\lambda)} + h_E) + 2(e_n^{(\lambda)\prime} + h_e{}')(e_n^{(\lambda)} + h_e)]|(\theta_{\mu,n+1}^{(\lambda)}) \\ &\qquad \lesssim O(n^{3-2\lambda})\sin^{2-2\lambda}\theta_{\mu,n+1}^{(\lambda)}. \end{aligned} \tag{3.24}$$

Finally, since

$$\begin{aligned}
&\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}\sin^{2\lambda-1}\theta\frac{d}{d\theta}P_n^{(\lambda)}(\cos\theta)\\
&\quad = 2[(e_n^{(\lambda)}+h_e)'[(E_{n+1}^{(\lambda)}+h_E)^2+(e_n^{(\lambda)}+h_e)^2]\\
&\qquad - (e_n^{(\lambda)}+h_e)[2(E_{n+1}^{(\lambda)}{}'+h_E{}')(E_{n+1}^{(\lambda)}+h_E)+2(e_n^{(\lambda)\prime}+h_e{}')(e_n^{(\lambda)}+h_e)]]\\
&\qquad \times([E_{n+1}^{(\lambda)}+h_E]^2+[e_n^{(\lambda)}+h_e]^2)^{-2}\\
&\qquad - \frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}(2\lambda-1)\sin^{2\lambda-2}\theta\cos\theta P_n^{(\lambda)}(\cos\theta),
\end{aligned}$$

we have by (3.24), (3.8), and (7.33.5) in [23],

$$\begin{aligned}
&\left|\sin^{2\lambda-1}\theta\frac{d}{d\theta}P_n^{(\lambda)}(\cos\theta)\right|_{\theta=\theta_{\mu,n+1}^{(\lambda)}}\\
&\quad\lesssim O(n^{3-2\lambda})\sin^{2-2\lambda}\theta_{\mu,n+1}^{(\lambda)}\times(O(n^{1-\lambda})\varphi^{1-\lambda}(\cos\theta_{\mu,n+1}^{(\lambda)}))^{-4}\\
&\qquad + |\sin^{2\lambda-2}\theta_{\mu,n+1}^{(\lambda)}\cos\theta_{\mu,n+1}^{(\lambda)}P_n^{(\lambda)}(\cos\theta_{\mu,n+1}^{(\lambda)})|\\
&\quad\lesssim O(n^{2\lambda-1})\varphi^{2\lambda-2}(\cos\theta_{\mu,n+1}^{(\lambda)})+O(n^{\lambda-1})\varphi^{\lambda-2}(\cos\theta_{\mu,n+1}^{(\lambda)})\\
&\quad\lesssim O(n^{2\lambda-1})\varphi^{2\lambda-2}(\cos\theta_{\mu,n+1}^{(\lambda)})
\end{aligned}$$

so that we have for $\mu=2,3,\ldots,n$,

$$\left|\frac{d}{d\theta}P_n^{(\lambda)}(\cos\theta)\right|_{\theta=\theta_{\mu,n+1}^{(\lambda)}}\lesssim n^{2\lambda-1}\varphi^{-1}(\cos\theta_{\mu,n+1}^{(\lambda)}).$$

Therefore, (3.15) is proved. □

Proof of Theorem 2.9. From (2.11) and (2.24), we have for $\mu=1,2,\ldots,n+1$,

$$\begin{aligned}
|E_{n+1}^{(\lambda)}{}''(\xi_{\mu,n+1}^{(\lambda)})| &\lesssim \varphi^{-2}(\xi_{\mu,n+1}^{(\lambda)})(|\xi_{\mu,n+1}^{(\lambda)}||E_{n+1}^{(\lambda)}{}'(\xi_{\mu,n+1}^{(\lambda)})|+n^2)\\
&\lesssim \varphi^{-2}(\xi_{\mu,n+1}^{(\lambda)})(n^{2-\lambda}\varphi^{-\lambda}(\xi_{\mu,n+1}^{(\lambda)})+n^2)\\
&\lesssim n^2\varphi^{-\lambda-2}(\xi_{\mu,n+1}^{(\lambda)}).
\end{aligned}$$

So, (2.26) is proved. Since by (3.13), (4.7.27), and (7.33.5) in [23], for $v=1,2,\ldots,n$,

$$\begin{aligned}
|E_{n+1}^{(\lambda)}{}'(x_{v,n}^{(\lambda)})P_n^{(\lambda)\prime}(x_{v,n}^{(\lambda)})| &\lesssim \frac{1}{\sin\phi_{v,n}^{(\lambda)}}\left|\frac{d}{d\theta}E_{n+1}^{(\lambda)}(\cos\phi_{v,n}^{(\lambda)})\right||P_n^{(\lambda)\prime}(x_{v,n}^{(\lambda)})|\\
&\lesssim n^{\lambda+1}\varphi^{-(\lambda+2)}(x_{v,n}^{(\lambda)})
\end{aligned}$$

and by (3.15) and (2.24) for $\mu = 1, 2, \dots, n+1$,

$$|E_{n+1}^{(\lambda)}{}'(\xi_{\mu,n+1}^{(\lambda)})P_n^{(\lambda)}{}'(\xi_{\mu,n+1}^{(\lambda)})| \lesssim |E_{n+1}^{(\lambda)}{}'(\xi_{\mu,n+1}^{(\lambda)})| \frac{1}{\sin \theta_{\mu,n+1}^{(\lambda)}} \left| \frac{d}{d\theta} P_n^{(\lambda)}(\cos \theta_{\mu,n+1}^{(\lambda)}) \right|$$
$$\lesssim n^{\lambda+1} \varphi^{-(\lambda+2)}(\xi_{\mu,n+1}^{(\lambda)}),$$

we have for $v = 1, 2, \dots, 2n+1$,

$$|E_{n+1}^{(\lambda)}{}'(y_{v,2n+1}^{(\lambda)})P_n^{(\lambda)}{}'(y_{v,2n+1}^{(\lambda)})| \lesssim n^{\lambda+1} \varphi^{-(\lambda+2)}(y_{v,2n+1}^{(\lambda)}). \tag{3.25}$$

Finally, since by (2.12), (3.11), (3.25), and (7.33.5) in [23],

$$|J_n(y_{v,2n+1}^{(\lambda)})| \lesssim \varphi^2(y_{v,2n+1}^{(\lambda)})|E_{n+1}^{(\lambda)}{}'(y_{v,2n+1}^{(\lambda)})||P_n^{(\lambda)}{}'(y_{v,2n+1}^{(\lambda)})|$$
$$+ |I_n(y_{v,2n+1}^{(\lambda)})||P_n^{(\lambda)}(y_{v,2n+1}^{(\lambda)})|$$
$$\lesssim n^{\lambda+1} \varphi^{-\lambda}(y_{v,2n+1}^{(\lambda)})$$

and from (2.18),

$$|F_{2n+1}^{(\lambda)}{}''(y_{v,2n+1}^{(\lambda)})| \lesssim \varphi^{-2}(y_{v,2n+1}^{(\lambda)})(|F_{2n+1}^{(\lambda)}{}'(y_{v,2n+1}^{(\lambda)})| + |J_n(y_{v,2n+1}^{(\lambda)})|),$$

we have by (2.25),

$$|F_{2n+1}^{(\lambda)}{}''(y_{v,2n+1}^{(\lambda)})| \lesssim n^{1+\lambda} \varphi^{-2-\lambda}(y_{v,2n+1}^{(\lambda)}).$$

Therefore, (2.27) is proved. □

Acknowledgments

This work was supported by Korea Research Foundation Grant(KRF-2002-050-C00003). The research on this project started when the author visited Prof. Sven Ehrich in GSF-IBB and the author thanks to Prof. Sven Ehrich for many kind suggestions and comments.

References

[1] P. Barrucand, Intégration Numérique, Abscisses de Kronrod-Patterson et Polynômes de Szegő, C.R. Acad. Sci. Paris, Ser. A 270 (1970) 147–158.

[2] L. Durand, Nichelson-type integrals for products of Gegenbauer functions and related topics, in: R. Askey (Ed.), Theory and Application of Special Functions, Academic Press, New York, 1975, pp. 353–374.

[3] S. Ehrich, Asymptotic properties of Stieltjes polynomials and Gauss–Kronrod quadrature formulae, J. Approx. Theory 82 (1995) 287–303.

[4] S. Ehrich, Stieltjes polynomials and the error of Gauss–Kronrod quadrature formulas, in: W. Gautschi, G. Golub, G. Opfer (Eds.), Applications and Computation of Orthogonal Polynomials, Proceedings of Conference on Oberwolfach, International Series Numerical Mathematics, Vol. 131, Birkhäuser, Basel, 1999, pp. 57–77.

[5] S. Ehrich, G. Mastroianni, On the generalized Stieltjes polynomials and Lagrange interpolation. Approximation Theory and Function Series (Budapest, 1995), 187–203, Bolyai Soc. Math. Stud., Vol. 5, János Bolyai Math. Soc., Budapest, 1996.
[6] S. Ehrich, G. Mastroianni, Stieltjes polynomials and Lagrange interpolation, Math. Comput. 66 (1997) 311–331.
[7] K.-J. Főrster, K. Petras, On estimates for the weights in Gaussian quadrature in the ultraspherical case, Math. Comput. 55 (1990) 243–264.
[8] W. Gautschi, Gauss–Kronrod quadrature—a survey, in: G.V. Milovanović (Ed.), Numerical Methods and Approximation Theory, Vol. III, (Niš, 1987), Univ. Niš, Niš, 1988, pp. 39–66.
[9] W. Gautschi, S.E. Notaris, An algebraic and numerical study of Gauss–Kronrod quadrature formulae for Jacobi weight functions, Math. Comput. 51 (1988) 231–248.
[10] J. Geronimus, On a set of polynomials, Ann. Math. 31 (1930) 681–686.
[11] A. Ghizzetti, A. Ossicini, Quadrature Formulae, Birkhäuser, Basel, 1970.
[12] H.S. Jung, Herimte and Hermite–Fejér interpolation polynomials for Stieltjes polynomials, preprint.
[13] G. Monegato, An overview of results and questions related to Kronrod schemes, in: G. Hämmerlin (Ed.), Numerishe Integration, Proceedings of Conference on Oberwolfach, International Series Numerical Mathematics, Vol. 24, Birkhaűser, Basel, 1979.
[14] G. Monegato, Some new inequalities related to certain ultraspherical polynomials, SIAM J. Math. Anal. 11 (1980) 663–667.
[15] G. Monegato, Stieltjes polynomials and related quadrature rules, SIAM Rev. 24 (1982) 137–158.
[16] NAG Fortran Library, Mark 15, NAG Ltd., Oxford, 1991.
[17] S.E. Notaris, An overview of results on the existence or nonexistence and the error term of Gauss–Kronrod quadrature formulae, in: R.V.M. Zahar (Ed.), Approximation and Computation, a Festschrift in Honor of Walter Gautschi, International Series Numerical Mathematics, Vol. 119, Birkhaűser, Basel, 1994, pp. 485–496.
[18] F. Peherstorfer, On the asymptotic behaviour of functions of the second kind and Stieltjes polynomials and on the Gauss–Kronrod quadrature formulae, J. Approx. Theory 70 (1992) 156–190.
[19] F. Peherstorfer, Stieltjes polynomials and functions of the second kind, J. Comput. Appl. Math. 65 (1995) 318–338.
[20] F. Peherstorfer, K. Petra, Ultraspherical Gauss–Kronrod quadrature is not possible for $\lambda > 3$, SIAM J. Numer. Anal. 37 (2000) 927–948.
[21] R. Piessens, E. de Doncker, C. Überhuber, D.K. Kahaner, QUADPACK—A Subroutine Package for Automatic Integration, in: Springer Series in Computational Mathematics, Vol. 1, Springer, Berlin, 1983.
[22] G. Szegő, Über gewisse orthogonale polynome, die zu einer oszillierenden Belegungsfunktion gehőren, Math. Ann. 110 (1934) 501–513.
[23] G. Szegő, Orthogonal polynomials, American Mathematical Society Colloquium Publications, Vol. 23, American Mathematical Society, Providence, RI, 1975.